Restoration of Limited Angle Tomography Data using Projection Onto Convex Sets

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Abstract

This paper investigates the restoration of limited angle tomography (LAT) data using projection operators to find a solution that is feasible given the data and prior information. We show how consistency can be enforced in the sinogram data for parallel beam geometries and why this method fails in the case of fan beam geometries.

1 Introduction

The goal of limited angle tomography as with all computerized tomography is to reconstruct an image of the internal structure of an object from projection data of the object. The need to reconstruct images where the data is limited in its angular range occurs in many applications of computed tomography. Data acquisition may be limited by physical obstructions, by time constraints or by radio opaque objects.

There are well established algorithms for solving computer tomography reconstructions when a complete data set is available, however these fail in the LAT case where there is insufficient data, see [1][9]. These efficient methods can be used if the missing data is first estimated.

Estimating or restoring the data prior to reconstructing the image is known as sinogram restoration, where the sinogram is the image of the projection data. Figure (2) shows the sinogram of the head phantom shown in figure (1).

The method of Projection onto Convex Sets is an algorithm that belongs to the Set Theoretic paradigm, see [4], that suggests one should estimate a solution that is feasible given the measured data and other known constraints on the solution rather than adopting some objective function that may require gross simplifications in order to make it’s solution computationally tractable. The conceptual basis for these methods are that certain known properties of the original image and measured data confine the image to lie in a convex set $S_i$. The intersection of these sets is the feasible region or set, denoted by $S$, in which the original image may lie.

$$S = \bigcap_{i \in I} S_i$$  \hspace{1cm} (1)

The method of projection onto convex sets uses projection operators to iteratively find a solution on the feasible region as can be see in equation (2) where $g_k$ is the current estimate of the solution and $P_i$ are projection operators. Projection operators project the current estimate of the solution onto the point of it’s associated convex set that is closest to the current estimate.

$$g_{k+1} = P_m P_{m-1} \ldots P_1 g_k$$  \hspace{1cm} (2)

POCS has been used in a number of image restoration applications [8] although only those that are useful for sinogram restoration are investigated here.

We wish to reconstruct an image $f(x, y)$ from parallel projection data given by the line integrals $g(r, \theta)$. For an object with spatial support over the unit circle line integrals can be written as follows [6].

$$g(r, \theta) = \int_{-\pi}^{\pi} f(r \cos \theta - s \sin \theta, r \sin \theta + s \cos \theta) ds$$  \hspace{1cm} (3)

When $g(r, \theta)$ is defined over the complete domain $\Gamma$ it is called a sinogram or the Radon transform of the object $f(x, y)$.

$$\Gamma = \{(r, \theta) | -1 \leq r \leq 1, 0 \leq \theta \leq 2\pi \}$$  \hspace{1cm} (4)
In LAT $g(r, \theta)$ is not measurable over the complete domain $\Gamma$ but only over $\Omega$, a subset of $\Gamma$ as is evident in figure (3).

$$\Omega = \{(r, \theta) | -1 \leq r \leq 1, 0 \leq \theta \leq \theta_L\} \cup \{(r, \theta) | -1 \leq r \leq 1, \pi \leq \theta \leq \pi + \theta_L\}$$  \hspace{1cm} (5)

2 Projection Operators

Here we will state a number of projection operators useful for sinogram restoration, for proofs see [5],[6].

2.1 Measurement Data

For an estimate of the sinogram that is consistent with the limited data that is available some constraint is needed that enforces data fidelity while taking into account the effects of noise. The measured data $g'$ provides values of the sinogram over the incomplete domain $\Omega$. We can put a constraint on $g$ given information about the noise.

$$g' = \begin{cases} g(r, \theta) + n(r, \theta) & (r, \theta) \in \Omega \\ 0 & (r, \theta) \notin \Omega \end{cases}$$  \hspace{1cm} (6)

Let $X_{\Omega}(r, \theta)$ be an indicator function for $\Omega$,

$$X_{\Omega}(r, \theta) = \begin{cases} 1 & (r, \theta) \in \Omega \\ 0 & (r, \theta) \notin \Omega \end{cases}$$  \hspace{1cm} (7)

then

$$C_3 = \{g | \|X_{\Omega}g - g'\| \leq \sigma\}$$  \hspace{1cm} (8)

This projection operator puts bounds on the deviation of the estimate from the measured data.

2.2 Sinogram Support

The sinogram support is that region in which the sinogram may be nonzero. Outside of this support the sinogram is zero. The sinogram support may be estimated from a LAT data set, see [10]. Let $\Lambda$ be the sinogram support of the
soidal and is given by equation (13), the associated projection operator \( P_2 \) is then given by

\[
P_2 g = \begin{cases} g(r, \theta) & (r, \theta) \in \Lambda \\ 0 & (r, \theta) \notin \Lambda \end{cases}
\] (10)

\[ a_M = \frac{1}{2\pi} \int_0^{2\pi} \int_{-1}^1 g(r, \theta) r^k \exp(i\theta) dr d\theta \] (16)

This simple nonlinear constraint demands that the estimate should be positive valued corresponding to X-ray absorption characteristics.

\[ C_4 = \{ g|g(r, \theta) \geq 0, \ (r, \theta) \in \Gamma \} \] (11)

The projection operator truncates negatively valued pixels to zero.

\[
P_3 g = \begin{cases} g & g(r, \theta) \geq 0 \\ 0 & g(r, \theta) < 0 \end{cases}
\] (12)

\[ a_M = 0, \quad |\theta| > k \] (17)

\[ g(r, \theta + \pi) = g(-r, \theta) \] (18)

In the Fourier domain this results in

\[ a_{k,l} = 0, \quad k + |l| : odd \] (19)

2.3 Nonnegativity

2.4 Sinogram Consistency

A sinogram is consistent if there exists a reconstruction that satisfies it. For a sinogram to be consistent it must satisfy a number of constraints. The projection data has a structure determined by the geometry of the imaging system. Each pixel in the image domain has a characteristic path in the Radon domain. For parallel projections the characteristic path of a point in the image plane on the sinogram is sinusoidal and is given by equation (13), \( R \) is the distance from the center of rotation to a point in the image domain. The Radon transform may be thought of as the superposition of these characteristic paths.

\[ r(\theta) = R \cos(\theta - \phi) \] (13)

The structure of a sinogram is governed by the Helgason-Ludwig consistency conditions which state that a sinogram may be decomposed into a number of moments \( A_k \) as shown below [6][9].

\[ a_k(\theta) = \int_{-1}^1 g(r, \theta) r^k dr \] (14)

where \( k \) is a positive integer. If we take the Fourier expansion of \( a_k(\theta) \) as

\[ a_k(\theta) = \sum_{-\infty}^{\infty} a_{M} \exp(j\theta) \] (15)

\[ b_M = \frac{1}{2\pi} \int_0^{2\pi} \int_{-1}^1 g(r, \theta) U_k(r) \exp(-j\theta) dr d\theta \] (20)

where \( U_k(r) \) is the \( k \) th order Chebyshev polynomial of the second kind defined by

\[ U_k(r) = \frac{\sin[(k + 1)\cos^{-1} r]}{\sin(\cos^{-1} r)} \] (21)

The inverse transform is then given by

\[ g(r, \theta) = \frac{1}{\pi} \sum_{k=0}^{\infty} \sum_{l=-\infty}^{\infty} b_{k,l} (1 - r^2)^{\frac{1}{2}} U_k(r) \exp(jl\theta) \] (22)

While the theory is correct for continuous functions, there are sources of error when using sampled data. The basis functions of equation (20) are no longer orthogonal once sampled. We are also limited in the number of terms of the Fourier series that we can take by sampling limits. Due
to the iterative nature of the POCS algorithm it is necessary to use orthogonal or unitary transforms. One is therefore forced to derive a transform matrix that is orthogonal and preserves the desired spectral properties or to adopt another orthogonal transform. For an introduction to unitary matrices see [7]. We have used the Discrete Cosine Transform (DCT) to decompose the sinogram into its moments as it produces a unitary transform matrix that is still close to the polynomial decomposition. However the spectral limits hold only approximately for the DCT. Let \( g(m, n) \) be a discrete estimate of \( g(r, \theta) \), then \( g(m, n) \) can be decomposed into coefficients \( c(k, l) \) by,

\[
e(k, l) = C g(m, n) F
\]  

(23)

where \( C \) is an \((M \times M)\) DCT matrix and \( F \) is an \((N \times N)\) DFT matrix. The projection operator to enforce symmetry can now be stated as follows,

\[
P_{\alpha} c(k, l) = \begin{cases} 
  c(k, l) & \text{k+l:even} \\
  0 & \text{k+l:odd}
\end{cases}
\]  

(24)

Finally the projection operator for the spectral limit constraint can be stated as

\[
P_{\beta} c(k, l) = \begin{cases} 
  c(k, l) & |l| \leq k \\
  0 & |l| > k
\end{cases}
\]  

(25)

3 Results

A simulated phantom was used for testing to make error estimates and comparisons to other methods possible. 50 projection slices were taken over an angular range of 100\(^{\circ}\). This incomplete data set can be seen in figure (3). A relative error measure was calculated as follows, where \( g_k \) is the estimated sinogram at the \( k \) th iteration.

\[
\text{error} = \frac{\|g - g_k\|_F}{\|g\|_F} \times 100\%
\]  

(26)

Figure 5: convergence of POCS with \( P_1, P_2, P_3, P_4, P_5 \)

Figure 6: restored sinogram

Now that a number of projection operators have been presented equation (2) can be used to estimate the missing data.
this respect the results are weaker than those that can be obtained by reconstructing using markov model priors to regularize the solution as in [2]. However due to its computational simplicity it may be desirable to use POCS to obtain a good initial estimate, that could be used by other iterative reconstruction algorithms as a good initial estimate is especially important for algorithms that are not globally convergent [3].

4 Fan beam geometries

The characteristic path generated by a fan beam geometry given by equation (27) and figure (9) is not sinusoidal and more importantly, it is not symmetric and changes shape for different points on the image plane. Therefore one cannot define a transformation of the data that will be highly spectrally limited. This is not to say that one cannot talk about fan beam data being inconsistent but just that one cannot enforce consistency by taking a unitary transform and enforcing some spectral support constraint. Although it is possible to rebin fan beam data, it would seem easier to apply these constraints by reconstructing the image directly from the incomplete data set possibly using ART, see [1][9], which is an example of a POCS reconstruction algorithm.

There are however benefits of preprocessing in the radon domain. The sinogram can be used to estimate the convex hull of the spatial support of the object from the sinogram support. This would be particularly useful for detecting whether part of the object fell outside the detectors field as this would lead to a local tomography problem that may cause the reconstruction algorithm to fail. Noise estimates can also be made in the sinogram domain which could then be used by the reconstruction algorithm.

5 Conclusions

We have presented a number of a priori constraints for estimating the missing data in limited-angle tomography data sets. While the methods were not strong enough to produce a unique solution they do provide means of obtaining a reasonable initial estimate which may be used by other reconstruction algorithms. We have also shown why these methods are not suitable for the restoration of fan beam data.
References


